ON THE ILL-POSEDNESS OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We prove the ill-posedness of three dimensional compressible viscous heat-conductive flows for the initial data belonging to the critical Besov space $(\dot B^{\frac{3}{p}}_{p,1}+\bar\rho,\, \dot B^{\frac{3}{p}-1}_{p,1},\, \dot B^{\frac{3}{p}-2}_{p,1})$ for $p>3$, here $\bar\rho$ is a positive constant. Especially, this result means that it seems impossible to construct a global solution for the highly oscillating initial velocity for the viscous heat-conductive flows. We also prove that the baratropic Navier-Stokes equation is ill-posed for the initial data belonging to the critical Besov space $(\dot B^{\frac{3}{p}}_{p,1}+\bar\rho,\, \dot B^{\frac{3}{p}-1}_{p,1})$ for $p>6$.

1. Introduction

We first consider the compressible viscous heat-conductive flows in $\mathbb{R}^+\times\mathbb{R}^3$:

$$\begin{cases}\partial_t\rho+\mathrm{div}(\rho u)=0,\\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mu\Delta u-(\lambda+\mu)\nabla\mathrm{div}u+\nabla P=0,\\ c_V(\partial_t(\rho\theta)+\mathrm{div}(\rho u\theta))-\kappa\Delta\theta+P\mathrm{div}u=\dfrac{\mu}{2}|\nabla u+(\nabla u)^\top|^2+\lambda|\mathrm{div}u|^2,\\ (\rho,u,\theta)|_{t=0}=(\rho_0,u_0,\theta_0).\end{cases}\tag{1.1}$$

Here $\rho(t,x)$, $u(t,x)$, $\theta(t,x)$ denote the density, velocity and temperature of the fluid respectively. The physical constants ν,λ are the viscosity coefficients satisfying

$$\mu>0\quad\text{and}\quad\lambda+2\mu>0,$$

and $c_V>0,\kappa>0$ are the specific heat at constant volume and thermal conductivity coefficient respectively. The pressure P is a function of ρ and θ. For simplicity, we restrict ourselves to the case of an ideal gas in which P takes the form

$$P=R\rho\theta,$$

for a universal constant $R>0$. For a matrix A, the notation $|A|^2$ denotes the trace of $AA^\top$, i.e.,

$$|A|^2=\mathrm{tr}(AA^\top)=\sum_{i,j}a_{ij}b_{ij}.$$

The local existence and uniqueness of smooth solution for the system (1.1) were proved by Nash [17] for smooth initial data without vacuum. Matsumura-Nishida[16] obtained the global well-posedness for smooth data close to equilibrium. For small initial data, the global existence of weak solutions was proved by Hoff [14]. For large initial data, Feireisl [10] proved the global existence of the variational solutions in the case of real gases. However, the global existence of weak solution and strong solution

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remains open, even in two dimensions. Recently, Sun-Wang-Zhang [18] proved a Beale-Kato-Majda type blow-up criterion for strong solution in terms of the upper bound of the density and temperature. For the initial density with compact support, Xin[19] proved that any non-zero smooth solution will blow up in the finite time.

Motived by Fujita-Kato's result [12] on the incompressible Navier-Stokes equations, Danchin studied in a series of papers [7, 8, 9] the well-posedness for the compressible Navier-Stokes equations in the critical spaces. Let us make it precise. It is easy to check that if (ρ, u, θ) is a solution of (1.1), then

$$(\rho_\lambda(t,x), u_\lambda(t,x), \theta_\lambda(t,x)) \stackrel{\text{def}}{=\!=} \big(\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2\theta(\lambda^2 t, \lambda x)\big),$$

is also a solution of (1.1) provided the pressure law has been changed into $\lambda^2 P$. A functional space is called critical if the associated norm is invariant under the transformation $(\rho, u, \theta) \longrightarrow (\rho_\lambda, u_\lambda, \theta_\lambda)$(up to a constant independent of λ). Then a natural candidate is the homogenous Sobolev space $\dot H^{\frac32} \times \big(\dot H^{\frac12}\big)^3 \times \dot H^{-\frac12}$. However, $\dot H^{\frac32}$ is not included in L^∞ such that one cannot expect to obtain a L^∞ control of the density when $\rho_0 - \bar\rho \in \dot H^{\frac32}$. Instead, one can choose the initial data (ρ_0, u_0, θ_0) such that for some $\bar\rho$,

$$(\rho_0 - \bar\rho, u_0, \theta_0) \in \dot B^{\frac3p}_{p,1} \times \big(\dot B^{\frac3p - 1}_{p,1}\big)^3 \times \dot B^{\frac3p-2}_{p,1}. \tag{1.2}$$

We refer to Definition 1.5 for Besov space.

In [8, 9], Danchin proved the global existence of (1.1) for small initial data in the critical Besov space as (1.2) with $p = 2$, and the local existence for general initial data in the critical Besov space with $p < 3$. For the baratropic Navier-Stokes equations, one can refer to [7] for the global existence with $p = 2$, and [5, 9] for the local existence with $p < 6$.

For the incompressible Navier-Stokes equations, Cannone et al. [2, 3] generalized Fujita-Kato's result to Besov spaces $\dot B^{-1+\frac3p}_{p,\infty}(p > 3)$ with negative regularity index. An important consequence of this result is that it allows to generate global solution for the highly oscillating initial velocity like

$$e^{\frac{x_3}{\epsilon}}(-\partial_2\phi(x), \partial_1\phi(x), 0),$$

since its norm is small in $\dot B^{-1+\frac3p}_{p,\infty}(p > 3)$ if ϵ is small, although it may be very large in the Sobolev space $\dot H^{\frac12}$. It is highly non-trival to generalize a similar result to the compressible Navier-Stokes system since it is a hyperbolic-parabolic coupled system. Very recently, important progress has been made by Chen-Miao-Zhang [6] and Charve-Danchin [4] where they construct the global solution for the highly oscillating initial velocity for the baratropic Navier-Stokes equations by proving the global well-posedness of the system (1.3) in the critical Besov space with $3 < p < 6$.

A natural question is whether a similar result remains true for the heat-conductive flows (1.1). The first step toward this problem is to prove a local existence result in the critical Besov space with $p > 3$ for the system (1.1). However, we prove that the system (1.1) is ill-posed in this case. More precisely, we prove

Theorem 1.1. Let $\bar\rho$ be a positive constant. Assume that $\rho_0 - \bar\rho \in \dot B^{3/p}_{p,1}$, $u_0 \in \dot B^{3/p-1}_{p,1}$, $\theta_0 \in \dot B^{3/p-2}_{p,1}$ for $p > 3$. Then the mapping $\mathbf{T} : (\rho_0, u_0, \theta_0) \longmapsto (\rho, u, \theta)$ is not C^2 continuous, where (ρ, u, θ) solves the system (1.1).

The mechanism leading to the ill-posedness comes from the high-high frequency interaction of the strong nonlinear terms $|\nabla u+(\nabla u)^\top|^2$ and $|\mathrm{div} u|^2$ in the temperature equation, which will behave very badly in the case when high-high frequency interaction evolves into a low frequency. Our proof is inspired by Germain's paper [13], where the author proved the ill-posedness of the incompressible Navier-Stokes equations in Besov space $\dot{B}^{-1}_{\infty,\infty}$(see also a different proof by Bourgain-Pavlović [1]).

We next consider the ill-posedness for the baratropic Navier-Stokes equations

$$\begin{cases} \partial_t\rho+\mathrm{div}(\rho u)=0, \\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mu\Delta u-(\lambda+\mu)\nabla \mathrm{div} u+\nabla P=0, \\ (\rho,u)|_{t=0}=(\rho_0,u_0). \end{cases} \tag{1.3}$$

Here the pressure P is a suitable smooth function of the density. We refer to the seminal books [15, 11] and reference therein for the related works.

In this case, the mechanism leading to the ill-posedness comes from the high-high frequency interaction of the nonlinear term $u\cdot\nabla u$. Since the nonlinear effect of $u\cdot\nabla u$ is weaker than that of $|\nabla u|^2$, the analysis is more delicate in order to capture the bad terms leading to the ill-posedness of (1.3). And the choice of the initial velocity is different from that used in the proof of Theorem 1.1.

Theorem 1.2. Let $\bar\rho$ be a positive constant. Assume that $\rho_0-\bar\rho\in\dot{B}^{3/p}_{p,1}$, $u_0\in\dot{B}^{3/p-1}_{p,1}$ for $p>6$. Then the mapping $\mathbf{T}:(\rho_0,u_0,\theta_0)\longmapsto(\rho,u,\theta)$ is not C^2 continuous, where (ρ,u) solves the system (1.3).

Remark 1.3. The well-posedness or ill-posedness remains open in the critical Besov spaces with $p=6$ for the baratropic Navier-Stokes system and $p=3$ for the heat-conductive flows.

Remark 1.4. We prove the ill-posedness of the compressible Navier-Stokes equations in the sense that the mapping $\mathbf{T}$ is not C^2 continuous. We conjecture that the system is also ill-posed in the sense of "norm inflation" in [1]. This is the goal of our future work.

Let us conclude the introduction by recalling the definition of Besov space. Choose a function $\varphi\in\mathcal{S}(\mathbb{R}^3)$ supported in $\mathcal{C}=\{\xi\in\mathbb{R}^3,\ \frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad\text{for all }\xi\neq 0.$$

The frequency localization operator Δ_j is defined by

$$\Delta_j f=\varphi(2^{-j}D)f\quad\text{for } j\in\mathbb{Z}.$$

We denote $\mathcal{Z}'(\mathbb{R}^3)$ by the space of the tempered space $\mathcal{S}'(\mathbb{R}^3)$ modulus the polynomial space $\mathcal{P}(\mathbb{R}^3)$.

Definition 1.5. Let $s\in\mathbb{R}$, $1\le p,q\le+\infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q}\stackrel{\mathrm{def}}{=}\big\{f\in\mathcal{Z}'(\mathbb{R}^3):\ \|f\|_{\dot{B}^s_{p,q}}<+\infty\big\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}}\stackrel{\mathrm{def}}{=}\Big\|2^{ks}\|\Delta_k f(t)\|_{L^p}\Big\|_{\ell^q}.$$

Notations. $\hat{f}$ and $\mathcal{F}f$ denote the Fourier transform of f, and $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f. The notation $A \cong B$ stands for $A = CB$ for a harmless constant C, and $A \sim B$ stands for $C_1 B \le A \le C_2 B$ for the harmless constants C_1, C_2. The summation convention over repeated indices is used.

2. Ill-posedness of the heat-conductive flows

This section is devoted to the proof of Theorem 1.1. Let $\delta > 0$ and (ρ, u, θ) be the solution of the following system:

$$(NS_\delta)\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mu\Delta u - (\lambda+\mu)\nabla\mathrm{div}u + R\nabla(\rho\theta) = 0, \\ c_V(\partial_t(\rho\theta) + \mathrm{div}(\rho u\theta)) - \kappa\Delta\theta + R\rho\theta\mathrm{div}u = \dfrac{\mu}{2}|\nabla u + (\nabla u)^T|^2 + \lambda|\mathrm{div}u|^2, \\ (\rho_0, u_0, \theta_0) = (\bar\rho + \delta\phi_\rho, \delta\phi_u, \delta\phi_\theta), \end{cases}$$

where $\phi_\rho, \phi_u, \phi_\theta$ will be determined later. By the uniqueness of the solution, we have

$$\big(\rho(\delta,x,t), u(\delta,x,t), \theta(\delta,x,t)\big)\Big|_{\delta=0} = (\bar\rho, 0, 0). \tag{2.1}$$

Set

$$(\rho', u', \theta') \triangleq \frac{d}{d\delta}(\rho, u, \theta)\Big|_{\delta=0}.$$

Taking the derivative with respect to δ on both sides of (NS_δ) and using (2.1) yield that

$$\begin{cases} \partial_t\rho' + \bar\rho\mathrm{div}u' = 0, \\ \bar\rho\partial_t u' - \mu\Delta u' - (\lambda+\mu)\nabla\mathrm{div}u' + R\bar\rho\nabla\theta' = 0, \\ c_V\bar\rho\partial_t\theta' - \kappa\Delta\theta' = 0, \\ (\rho_0', u_0', \theta_0') = (\phi_\rho, \phi_u, \phi_\theta). \end{cases} \tag{2.2}$$

We define

$$\Lambda^s f \triangleq \mathcal{F}^{-1}(|\xi|^s\hat{f}(\xi)) \quad \text{for} \quad s\in\mathbb{R}.$$

For a vector u, let us denote $\widetilde{\mathrm{curl}}u$ by

$$(\widetilde{\mathrm{curl}}u)_j^\ell = \partial_j u^\ell - \partial_\ell u^j.$$

Applying the operator $\Lambda^{-1}\mathrm{div}$ and $\Lambda^{-1}\widetilde{\mathrm{curl}}$ to the second equation of (2.2) respectively, and noting that

$$u' = -\Lambda^{-1}\nabla h' - \Lambda^{-1}\mathrm{div}\Omega'$$

with $h' = \Lambda^{-1}\mathrm{div}u'$ and $\Omega' = \Lambda^{-1}\widetilde{\mathrm{curl}}u'$, we deduce that

$$\begin{cases} \rho'(t) = \phi_\rho - \bar\rho\displaystyle\int_0^t \mathrm{div}u' d\tau, \\ u'(t) = \Phi(\phi_u) - R\displaystyle\int_0^t e^{\bar\nu(t-\tau)\Delta}\nabla\theta'(\tau)d\tau, \\ \theta'(t) = e^{\bar\kappa\Delta t}\phi_\theta, \end{cases}$$

where

$$\Phi(\phi_u) \triangleq -e^{\bar\nu t\Delta}\Lambda^{-2}\nabla\mathrm{div}\phi_u - e^{\bar\mu t\Delta}\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}\phi_u,$$

and $\bar{\kappa} = \kappa/(\bar{\rho}c_V)$, $\bar{\mu} = \mu/\bar{\rho}$, $\bar{\lambda} = \lambda/\bar{\rho}$, $\bar{\nu} = \bar{\lambda} + 2\bar{\mu}$. So, if $\bar{\kappa} \neq \bar{\nu}$, we get

$$\begin{cases} \rho'(t) = \phi_\rho - \bar{\rho}\Big(\dfrac{e^{\bar{\nu}t\Delta} - I}{\bar{\nu}\Delta}\Big)\mathrm{div}\phi_u + \dfrac{R\bar{\rho}}{\bar{\kappa} - \bar{\nu}}\Big(\dfrac{e^{\bar{\kappa}t\Delta} - I}{\bar{\kappa}\Delta} - \dfrac{e^{\bar{\nu}t\Delta} - I}{\bar{\nu}\Delta}\Big)\phi_\theta, \\ u'(t) = \Phi(\phi_u) - \dfrac{R}{\bar{\kappa} - \bar{\nu}}\Big(\dfrac{e^{\bar{\kappa}t\Delta} - e^{\bar{\nu}t\Delta}}{\Delta}\Big)\nabla\phi_\theta, \\ \theta'(t) = e^{\bar{\kappa}\Delta t}\phi_\theta, \end{cases} \tag{2.3}$$

and if $\bar{\kappa} = \bar{\nu}$, we get

$$\begin{cases} \rho'(t) = \phi_\rho - \bar{\rho}\Big(\dfrac{e^{\bar{\nu}t\Delta} - I}{\bar{\nu}\Delta}\Big)\mathrm{div}\phi_u + \dfrac{R\bar{\rho}}{\bar{\nu}}e^{\bar{\nu}t\Delta}t\phi_\theta - \dfrac{R\bar{\rho}}{\bar{\nu}^2}\Big(\dfrac{e^{\bar{\nu}t\Delta} - I}{\Delta}\Big)\phi_\theta, \\ u'(t) = \Phi(\phi_u) - Re^{\bar{\nu}\Delta t}t\nabla\phi_\theta, \\ \theta'(t) = e^{\bar{\kappa}\Delta t}\phi_\theta. \end{cases} \tag{2.4}$$

Set

$$(\rho'', u'', \theta'') \triangleq \frac{d^2}{d\delta^2}(\rho, u, \theta)\Big|_{\delta=0}.$$

Taking the second order derivative with respect to δ on both sides of (NS_δ) and thanks to (2.1), we obtain

$$\begin{cases} \partial_t\rho'' + 2\rho'\mathrm{div}u' + \bar{\rho}\mathrm{div}u'' + 2\nabla\rho' u' = 0, \\ \bar{\rho}\partial_t u'' + 2\rho'\partial_t u' + 2\bar{\rho}u' \cdot \nabla u' - \mu\Delta u'' - (\lambda + \mu)\nabla\mathrm{div}u'' + 2R\nabla\rho' \cdot \theta' \\ \qquad + 2R\rho' \cdot \nabla\theta' + R\bar{\rho}\nabla\theta'' = 0, \\ c_V(\bar{\rho}\partial_t\theta'' + 2\rho'\partial_t\theta' + 2\bar{\rho}u' \cdot \nabla\theta') - \kappa\Delta\theta'' = \mu|\nabla u' + (\nabla u')^\top|^2 + 2\lambda|\mathrm{div}u'|^2 \\ \qquad - 2R\bar{\rho}\theta'\mathrm{div}u', \\ (\rho_0'', u_0'', \theta_0'') = (0, 0, 0). \end{cases}$$

Especially, we have by (2.2) that

$$\begin{aligned} \partial_t\theta'' - \bar{\kappa}\Delta\theta'' &= \frac{\bar{\mu}}{c_V}|\nabla u' + (\nabla u')^\top|^2 + \frac{2\bar{\lambda}}{c_V}|\mathrm{div}u'|^2 - \frac{2R}{c_V}\theta'\mathrm{div}u' - \frac{2\bar{\kappa}}{\bar{\rho}}\rho'\Delta\theta' - 2u'\nabla\theta' \\ &\triangleq F_1 + F_2 + F_3 + F_4 + F_5. \end{aligned}$$

Hence, we get

$$\theta''(t) = \sum_{J=1}^{5}\int_0^t e^{\bar{\kappa}(t-\tau)\Delta}F_J d\tau \triangleq \sum_{J=1}^{5}\mathfrak{G}_J. \tag{2.5}$$

Let ϕ be a smooth, radial, non-negative function in $\mathbb{R}^3$ such that $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$. We define ϕ_u by its Fourier transform

$$\widehat{\phi_u}(\xi) = \bigg(\sum_{k=10}^{N}\alpha_k 2^{(1-\frac{3}{p})k}\phi(\xi - 2^k e_1) + \sum_{k=10}^{N}\alpha_k 2^{(1-\frac{3}{p})k}\phi(\xi + 2^k e_1),\ 0,\ 0\bigg), \tag{2.6}$$

where $e_1 = (1, 0, 0)$ and $\{\alpha_k\}$ is a series belonging to ℓ^1 such that for any fixed $\varepsilon_0 > 0$, $\sum_{k=1}^{\infty}\alpha_k^2 2^{\varepsilon_0 k} = \infty$. It is easy to check that ϕ_u is uniformly bounded in $\dot{B}_{p,1}^{\frac{3}{p}-1}$. We choose $\phi_\rho, \phi_\theta \in \mathcal{S}(\mathbb{R}^3)$ such that

$$\widehat{\phi_\rho}(\xi) = \widehat{\phi_\theta}(\xi) = \overline{\phi}(\xi),$$

where $\overline{\phi}(\xi) \in \mathcal{S}(\mathbb{R}^3)$ is supported in $\{\xi : 1 \leq |\xi| \leq 2\}$.

Throughout this section, we fix a vector $\xi_0 = (0, 1/10, 1/10)$ and take $\epsilon > 0$ small enough.

Case 1. $\bar\kappa \neq \bar\nu$

- The estimates of $\mathfrak{G}_3, \mathfrak{G}_4, \mathfrak{G}_5$.

Due to (2.3), we have

$$\begin{aligned}\mathfrak{G}_3 = \int_0^t e^{\bar\kappa(t-\tau)\Delta} F_3 d\tau \cong \int_0^t & e^{\bar\kappa(t-\tau)\Delta}(e^{\bar\kappa\tau\Delta}\phi_\theta e^{\bar\nu\tau\Delta}\mathrm{div}\phi_u)\\ & - e^{\bar\kappa(t-\tau)\Delta}\big\{e^{\bar\kappa\tau\Delta}\phi_\theta(e^{\bar\kappa\tau\Delta} - e^{\bar\nu\tau\Delta})\phi_\theta\big\}d\tau \triangleq \mathfrak{G}_{31} + \mathfrak{G}_{32},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{G}_4 = &\int_0^t e^{\bar\kappa(t-\tau)\Delta} F_4 d\tau \cong \int_0^t e^{\bar\kappa(t-\tau)\Delta}\Big\{(I - e^{\bar\nu\tau\Delta})\Delta^{-1}\mathrm{div}\phi_u e^{\bar\kappa\Delta\tau}\Delta\phi_\theta\Big\}\\ & + e^{\bar\kappa(t-\tau)\Delta}\Big\{\phi_\rho e^{\bar\kappa\Delta\tau}\Delta\phi_\theta + \Big(\frac{e^{\bar\kappa\tau\Delta} - e^{\bar\nu\tau\Delta}}{\Delta}\Big)\phi_\theta e^{\bar\kappa\Delta\tau}\Delta\phi_\theta\Big\}d\tau\\ \triangleq & \mathfrak{G}_{41} + \mathfrak{G}_{42},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{G}_5 = \int_0^t e^{\bar\kappa(t-\tau)\Delta} F_5 d\tau \cong & \int_0^t e^{\bar\kappa(t-\tau)\Delta}\big\{\Phi(\phi_u)e^{\bar\kappa\Delta\tau}\nabla\phi_\theta\big\}\\ & - e^{\bar\kappa(t-\tau)\Delta}\Big\{\Big(\frac{e^{\bar\kappa\tau\Delta} - e^{\bar\nu\tau\Delta}}{\Delta}\Big)\nabla\phi_\theta e^{\bar\kappa\Delta\tau}\nabla\phi_\theta\Big\}d\tau \triangleq \mathfrak{G}_{51} + \mathfrak{G}_{52}.\end{aligned}$$

Thanks to the choice of $\phi_\theta, \phi_\rho, \phi_u$ and ξ_0, it is easy to check that

$$\begin{aligned}&\widehat{\mathfrak{G}}_{31}(t,\xi) = \widehat{\mathfrak{G}}_{41}(t,\xi) = \widehat{\mathfrak{G}}_{51}(t,\xi) = 0 \quad \text{for} \quad \xi \in B(\xi_0, \epsilon),\\ &\mathfrak{G}_{32} \in \mathcal{S}(\mathbb{R}^3), \quad \mathfrak{G}_{42},\ \mathfrak{G}_{52} \in L^2(\mathbb{R}^3).\end{aligned} \tag{2.7}$$

- The estimate of $\mathfrak{G}_2$.

Plugging (2.3) into the term F_2, we get

$$\begin{aligned}\mathfrak{G}_2 \cong & \int_0^t e^{\bar\kappa(t-\tau)\Delta}(e^{\bar\nu t\Delta}\mathrm{div}\phi_u)^2 d\tau + \int_0^t e^{\bar\kappa(t-\tau)\Delta}\big\{(e^{\bar\kappa\tau\Delta}\phi_\theta)^2 - 2e^{\bar\kappa\tau\Delta}\phi_\theta e^{\bar\nu\tau\Delta}\phi_\theta\\ & + (e^{\bar\nu\tau\Delta}\phi_\theta)^2\big\}d\tau + \int_0^t e^{\bar\kappa(t-\tau)\Delta}\big\{e^{\bar\nu\tau\Delta}\mathrm{div}\phi_u e^{\bar\nu\tau\Delta}\phi_\theta - e^{\bar\nu\tau\Delta}\mathrm{div}\phi_u e^{\bar\kappa\tau\Delta}\phi_\theta\big\}d\tau\\ \triangleq & \mathfrak{G}_{21} + \mathfrak{G}_{22} + \mathfrak{G}_{23}.\end{aligned}$$

Let us first calculate Fourier transform of $\mathfrak{G}_{21}$ as follows

$$\begin{aligned}\widehat{\mathfrak{G}}_{21}(t,\xi) &= \int_0^t e^{-\bar\kappa|\xi|^2(t-\tau)}\mathcal{F}((e^{\nu\tau\Delta}\mathrm{div}\phi_u)^2)(\xi)d\tau\\ &= -e^{-\bar\kappa|\xi|^2 t}\int_0^t\int_{\mathbb{R}^3} e^{(\bar\kappa|\xi|^2 - \bar\nu|\xi-\eta|^2 - \bar\nu|\eta|^2)\tau}(\xi-\eta)_j\widehat{\phi_u^j}(\xi-\eta)\eta_\ell\widehat{\phi_u^\ell}(\eta)d\eta d\tau.\end{aligned}$$

Due to the definition of ϕ_u, for $\xi \in B(\xi_0, \epsilon)$, $\widehat{\mathfrak{G}}_{21}(t,\xi)$ equals to

$$-e^{-\bar{\kappa}|\xi|^2 t}\int_{\mathbb{R}^3}\frac{e^{(\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta|^2-\bar{\nu}|\eta|^2)t}-1}{\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta|^2-\bar{\nu}|\eta|^2}(\xi_1-\eta_1)\eta_1$$
$$\times\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}\phi(\xi-\eta+2^k e_1)\phi(\eta-2^k e_1)d\eta \triangleq \widehat{\mathfrak{G}}_{21}^1(t,\xi)$$

plus

$$-e^{-\bar{\kappa}|\xi|^2 t}\int_{\mathbb{R}^3}\frac{e^{(\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta|^2-\bar{\nu}|\eta|^2)t}-1}{\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta|^2-\bar{\nu}|\eta|^2}(\xi_1-\eta_1)\eta_1$$
$$\times\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}\phi(\xi-\eta-2^k e_1)\phi(\eta+2^k e_1)d\eta \triangleq \widehat{\mathfrak{G}_{21}^2}(t,\xi).$$

Making the change of variables, the term $\widehat{\mathfrak{G}}_{21}^1(t,\xi)$ turns into

$$-e^{-\bar{\kappa}|\xi|^2 t}\int_{\mathbb{R}^3}\sum_{k=10}^{N}\frac{e^{(\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta-2^k e_1|^2-\bar{\nu}|\eta+2^k e_1|^2)t}-1}{\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta-2^k e_1|^2-\bar{\nu}|\eta+2^k e_1|^2}(\xi_1-\eta_1-2^k)(\eta_1+2^k)$$
$$\times\alpha_k^2 2^{2(1-\frac{3}{p})k}\phi(\xi-\eta)\phi(\eta)d\eta.$$

Hence, if $\xi \in B(\xi_0,\epsilon),\ t=\frac{1}{2^{20}}$, it is easy to see that

$$\widehat{\mathfrak{G}}_{21}(t,\xi)\sim\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}. \tag{2.8}$$

For $\mathfrak{G}_{23}$, we have

$$\mathfrak{G}_{23}=\frac{R}{(\bar{\kappa}-\bar{\nu})}\int_0^t e^{\bar{\kappa}(t-\tau)\Delta}(e^{\bar{\nu}\tau\Delta}\mathrm{div}\phi_u(e^{\bar{\nu}\tau\Delta}-e^{\bar{\kappa}\tau\Delta})\phi_\theta)d\tau.$$

Thanks to the choice of ϕ_u,ϕ_θ and ξ_0, we find that for $\xi\in B(\xi_0,\epsilon)$,

$$\widehat{\mathfrak{G}}_{23}(t,\xi)=0 \quad \text{and} \quad \mathfrak{G}_{22}\in\mathcal{S}(\mathbb{R}^3). \tag{2.9}$$

- The estimate of $\mathfrak{G}_1$.

Thanks to (2.3), we have

$$F_1\cong\big|A+B\big|^2,$$

where the matrix A, B is given by

$$A=\nabla\Phi(\phi_u)+(\nabla\Phi(\phi_u))^\top,$$
$$B=-\nabla^2\Delta^{-1}(e^{\bar{\kappa}t\Delta}-e^{\bar{\nu}t\Delta})\phi_\theta-\big(\nabla^2\Delta^{-1}(e^{\bar{\kappa}t\Delta}-e^{\bar{\nu}t\Delta})\phi_\theta\big)^\top.$$

Hence we get

$$\mathfrak{G}_1=\int_0^t e^{\bar{\kappa}(t-\tau)\Delta}F_1 d\tau\cong\sum_{\ell,m=1}^{3}\int_0^t e^{\bar{\kappa}(t-\tau)\Delta}(A_{\ell m}^2+2A_{\ell m}B_{\ell m}+B_{\ell m}^2)d\tau$$
$$\triangleq\mathfrak{G}_{11}+\mathfrak{G}_{12}+\mathfrak{G}_{13},$$

where $A_{\ell m}$ and $B_{\ell m}$ are the elements of the matrix A and B respectively. By the choice of ϕ_u,ϕ_θ and ξ_0, we have

$$\widehat{\mathfrak{G}}_{12}(t,\xi)=0 \quad for \quad \xi\in B(\xi_0,\epsilon) \quad \text{and} \quad \mathfrak{G}_{13}\in\mathcal{S}(\mathbb{R}^3). \tag{2.10}$$

Now we turn to the trouble term $\mathfrak{G}_{11}$. The Fourier transform of $A_{\ell m}$ is given by

$$\begin{aligned}
&2i\frac{e^{-\bar{\nu}t|\eta|^2}}{|\eta|^2}\eta_\ell\eta_m\eta_j\widehat{\phi_u^j}+2i\frac{e^{-\bar{\mu}t|\eta|^2}}{|\eta|^2}\eta_\ell\eta_j\eta_m\widehat{\phi_u^j}\\
&\qquad -i\frac{e^{-\bar{\mu}t|\eta|^2}}{|\eta|^2}\eta_j\eta_j\big(\eta_\ell\widehat{\phi_u^m}+\eta_m\widehat{\phi_u^\ell}\big)\triangleq\sum_{J=1}^{3}\widehat{A}_{\ell m}^J(t,\eta).
\end{aligned}$$

Hence,

$$\begin{aligned}
\widehat{\mathfrak{G}}_{11}(t,\xi)=&\sum_{J,J'=1}^{3}\sum_{\ell,m=1}^{3}\int_0^t\!\!\int_{\mathbb{R}^3}e^{-\bar{\kappa}|\xi|^2(t-\tau)}\widehat{A}_{\ell m}^J(\xi-\eta)\widehat{A}_{\ell m}^{J'}(\eta)d\eta d\tau\\
\triangleq&\sum_{J,J'=1}^{3}\mathfrak{A}^{JJ'}(t,\xi).
\end{aligned}$$

First of all, we have

$$\begin{aligned}
\mathfrak{A}^{12}+\mathfrak{A}^{13}=&\int_0^t\!\!\int_{\mathbb{R}^3}e^{-\bar{\kappa}|\xi|^2(t-\tau)}\widehat{A}_{\ell m}^1(\xi-\eta)\{\widehat{A}_{\ell m}^2(\eta)+\widehat{A}_{\ell m}^3(\eta)\}d\eta d\tau\\
=2e^{-\bar{\kappa}|\xi|^2t}&\int_0^t\!\!\int_{\mathbb{R}^3}\frac{e^{(\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta|^2-\bar{\mu}|\eta|^2)\tau}}{|\xi-\eta|^2|\eta|^2}(\xi-\eta)_\ell(\xi-\eta)_m\\
&\times(\xi-\eta)_j\widehat{\phi_u^j}(\xi-\eta)(\eta_\ell\eta_{j'}\eta_{j'}\widehat{\phi_u^m}(\eta)-\eta_\ell\eta_{j'}\eta_m\widehat{\phi_u^{j'}}(\eta))d\eta d\tau.
\end{aligned}$$

Recalling the choice of $\widehat{\phi}_u$ and making change of variables, $\mathfrak{A}^{12}(t,\xi)+\mathfrak{A}^{13}(t,\xi)$ for $\xi\in B(\xi_0,\epsilon)$ equals to

$$\begin{aligned}
&-e^{-\bar{\kappa}|\xi|^2t}\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}\int_{\mathbb{R}^3}\frac{e^{(\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta-2^ke_1|^2-\bar{\mu}|\eta+2^ke_1|^2)t}-1}{\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta-2^ke_1|^2-\bar{\mu}|\eta+2^ke_1|^2}\phi(\xi-\eta)\phi(\eta)\\
&\quad\times\frac{(\xi-\eta-2^ke_1)_\ell}{|\xi-\eta-2^ke_1|^2}(\xi-\eta-2^ke_1)_1\frac{(\eta+2^ke_1)_\ell}{|\eta+2^ke_1|^2}\Big\{(\xi-\eta-2^ke_1)_1(\eta+2^ke_1)_{j'}\\
&\quad\times(\eta+2^ke_1)_{j'}-(\xi-\eta-2^ke_1)_m(\eta+2^ke_1)_1(\eta+2^ke_1)_m\Big\}d\eta
\end{aligned}$$

plus a similar term corresponding to $\phi(\xi-\eta-2^ke_1)\phi(\eta+2^ke_1)$. Then for $\xi\in B(\xi_0,\epsilon)$ and $t=\frac{1}{2^{20}}$, we have

$$\big|\mathfrak{A}^{12}(t,\xi)+\mathfrak{A}^{13}(t,\xi)\big|=\big|\mathfrak{A}^{21}(t,\xi)+\mathfrak{A}^{31}(t,\xi)\big|\sim\sum_{k=10}^{N}\alpha_k^2 2^{-\frac{6}{p}k}. \tag{2.11}$$

Similarly we have

$$\big|\mathfrak{A}^{22}(t,\xi)+\mathfrak{A}^{23}(t,\xi)\big|,\ \big|\mathfrak{A}^{32}(t,\xi)+\mathfrak{A}^{33}(t,\xi)\big|\sim\sum_{k=10}^{N}\alpha_k^2 2^{-\frac{6}{p}k}, \tag{2.12}$$

for $\xi\in B(\xi_0,\epsilon)$ and $t=\frac{1}{2^{20}}$. On the other hand,

$$\begin{aligned}\mathfrak{A}^{11}(t,\xi)=&-2e^{-\bar{\kappa}|\xi|^2t}\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}\int_{\mathbb{R}^3}\frac{e^{(\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta-2^ke_1|^2-\bar{\nu}|\eta+2^ke_1|^2)t}-1}{\bar{\kappa}|\xi|^2-\bar{\nu}|\xi-\eta-2^ke_1|^2-\bar{\nu}|\eta+2^ke_1|^2}\\&\times\frac{(\xi-\eta-2^ke_1)_\ell}{|\xi-\eta-2^ke_1|^2}\frac{(\eta+2^ke_1)_\ell}{|\eta+2^ke_1|^2}(\xi-\eta-2^ke_1)_m(\eta+2^ke_1)_m\\&\times(\xi-\eta-2^ke_1)_1(\eta+2^ke_1)_1\phi(\xi-\eta)\phi(\eta)d\eta,\end{aligned}$$

plus a similar term corresponding to $\phi(\xi-\eta-2^ke_1)\phi(\eta+2^ke_1)$. It is easy to deduce that for $\xi\in B(\xi_0,\epsilon)$ and $t=\frac{1}{2^{20}}$,

$$\mathfrak{A}^{11}(t,\xi)\sim\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}. \tag{2.13}$$

Summing up (2.11)-(2.13), we obtain

$$\widehat{\mathfrak{G}}_{11}(t,\xi)\geq C\Big(\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}-\sum_{k=10}^{N}\alpha_k^2 2^{-\frac{6}{p}k}\Big), \tag{2.14}$$

for $\xi\in B(\xi_0,\epsilon)$ and $t=\frac{1}{2^{20}}$.

Collecting (2.7)-(2.10) and (2.14) together, we show that for $g\in\mathcal{S}(\mathbb{R}^3)$ with $\hat{g}>0$ and supported in the ball $B(\xi_0,\epsilon)$,

$$\big\langle\theta'',\ g\big\rangle=\big\langle\sum_{1\leq J\leq 5}\widehat{\mathfrak{G}}_J\,,\ \widehat{g}\big\rangle\geq C\sum_{k=10}^{N}\alpha_k^2 2^{2(1-\frac{3}{p})k}-C.$$

This implies that θ'' is not bounded in $\mathcal{S}'(\mathbb{R}^3)$ due to $p>3$. Thus, $D^2\mathbf{T}$ is unbounded from $\big(\dot{B}^{\frac{3}{p}}_{p,1}\times\dot{B}^{\frac{3}{p}-1}_{p,1}\times\dot{B}^{\frac{3}{p}-2}_{p,1}\big)\times\big(\dot{B}^{\frac{3}{p}}_{p,1}\times\dot{B}^{\frac{3}{p}-1}_{p,1}\times\dot{B}^{\frac{3}{p}-2}_{p,1}\big)$ to $\mathcal{S}'(\mathbb{R}^3)$.

Case 2. $\bar{\kappa}=\bar{\nu}$

Let us return to (2.5). Plugging (2.4) into the term F, we get

$$\begin{aligned}F_1=&\frac{\bar{\mu}}{c_V}|\nabla\Phi(\phi_u)+(\nabla\Phi(\phi_u))^\top-Re^{\bar{\nu}t\Delta}t(\nabla^2\phi_\theta+(\nabla^2\phi_\theta)^\top)|^2,\\F_2=&\frac{2\bar{\lambda}}{c_V}\big((e^{\bar{\nu}t\Delta}\mathrm{div}\phi_u)^2+(Rte^{\bar{\nu}t\Delta}\Delta\phi_\theta)^2-2Rte^{\bar{\nu}t\Delta}\mathrm{div}\phi_u e^{\bar{\nu}t\Delta}\Delta\phi_\theta\big),\\F_3=&\frac{2R}{c_V}\big(e^{\bar{\kappa}t\Delta}\phi_\theta e^{\bar{\nu}t\Delta}\mathrm{div}\phi_u-Rte^{\bar{\kappa}t\Delta}\phi_\theta e^{\bar{\nu}t\Delta}\Delta\phi_\theta\big),\\F_4=&\frac{2\bar{\kappa}}{\bar{\rho}}\phi_\rho e^{\bar{\kappa}t\Delta}\Delta\phi_\theta-2\bar{\kappa}\Big(\frac{e^{\bar{\nu}t\Delta}-I}{\bar{\nu}\Delta}\Big)\mathrm{div}\phi_u e^{\bar{\kappa}t\Delta}\Delta\phi_\theta\\&+\frac{2\bar{\kappa}R}{\bar{\nu}}te^{\bar{\nu}\Delta}\phi_\theta e^{\bar{\kappa}t\Delta}\Delta\phi_\theta-\frac{R\bar{\rho}}{\bar{\nu}^2}\Big(\frac{e^{\bar{\nu}t\Delta}-I}{\Delta}\Big)\phi_\theta e^{\bar{\kappa}t\Delta}\Delta\phi_\theta,\\F_5=&2\Phi(\phi_u)e^{\bar{\kappa}t\Delta}\nabla\phi_\theta-2Rte^{\bar{\nu}t\Delta}\nabla\phi_\theta e^{\bar{\kappa}t\Delta}\nabla\phi_\theta,\end{aligned}$$

Similar to the case $\bar{\kappa}\neq\bar{\nu}$, the term

$$\int_0^t e^{\bar{\kappa}(t-\tau)\Delta}\Big(\big|\nabla\Phi(\phi_u)+(\nabla\Phi(\phi_u))^\top\big|^2+(e^{\bar{\nu}\tau\Delta}\mathrm{div}\phi_u)^2\Big)d\tau$$

is not bounded in $\mathcal{S}'(\mathbb{R}^3)$, while the other terms are bounded in $\mathcal{S}'(\mathbb{R}^3)$. Then θ'' is also not bounded in $\mathcal{S}'(\mathbb{R}^3)$ in the case of $\bar{\kappa} = \bar{\nu}$.

3. Ill-posedness of the baratropic Navier-Stokes equations

This section is devoted to the proof of Theorem 1.2. With the similar notations in Section 2, (ρ', u') satisfies

$$\begin{cases} \partial_t \rho' + \bar{\rho}\mathrm{div} u' = 0, \\ \bar{\rho}\partial_t u' - \mu\Delta u' - (\lambda+\mu)\nabla \mathrm{div} u' + P'(\bar{\rho})\nabla\rho' = 0, \\ (\rho'(x,0),\, u'(x,0)) = (\phi_\rho(x), \phi_u(x)), \end{cases}$$

which can be rewritten as

$$\begin{cases} \partial_t^2 \rho' - \nu\Delta\partial_t\rho' - P'(\bar{\rho})\Delta\rho' = 0, \\ \partial_t \Lambda h' - \bar{\nu}\Delta\Lambda h' + \bar{\rho}^{-1}P'(\bar{\rho})\Delta\rho' = 0, \\ \partial_t \Omega' - \bar{\mu}\Delta\Omega' = 0, \\ (\rho'(x,0),\, u'(x,0)) = (\phi_\rho(x), \phi_u(x)), \\ \partial_t\rho'(x,0) = -\bar{\rho}\mathrm{div}\phi_u(x), \end{cases}$$

here $h' = \Lambda^{-1}\mathrm{div} u'$ and $\Omega' = \Lambda^{-1}\widetilde{\mathrm{curl}} u'$ (note that $(\widetilde{\mathrm{curl}} u)_j^\ell = \partial_j u^\ell - \partial_\ell u^j$, so here Ω' is a matrix). Thanks to Lemma 4.1 in [6], we know that

$$\begin{pmatrix} \widehat{\rho'} \\ \widehat{h'} \end{pmatrix} = \begin{bmatrix} \widetilde{K}(t,\xi) & -\bar{\rho}L(t,\xi)|\xi| \\ \frac{\tilde{P}}{\bar{\rho}}L(t,\xi)|\xi| & K(t,\xi) \end{bmatrix} \begin{pmatrix} \widehat{\phi}_\rho(\xi) \\ i|\xi|^{-1}\xi\cdot\widehat{\phi}_u(\xi) \end{pmatrix},$$

where

$$\widetilde{K}(t,\xi) = \frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+ t}}{\lambda_+(\xi)-\lambda_-(\xi)}, \quad L(t,\xi) = \frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)},$$
$$K(t,\xi) = \frac{\lambda_+(\xi)e^{\lambda_+(\xi)t} - \lambda_-(\xi)e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)},$$

with

$$\lambda_\pm(\xi) = -\frac{1}{2}\nu|\xi|^2 \pm \frac{1}{2}\sqrt{\nu^2|\xi|^4 - 4\tilde{P}|\xi|^2} \quad \text{with } \tilde{P} \triangleq P'(\bar{\rho}).$$

Then we have

$$\begin{pmatrix} \widehat{\rho'} \\ \widehat{h'} \end{pmatrix} = \widehat{\mathcal{G}}(t,\xi)\begin{pmatrix} \widehat{\phi}_\rho(\xi) \\ \widehat{\phi}_u(\xi) \end{pmatrix}, \quad \Omega'(t,x) = e^{\bar{\mu}\Delta t}\Lambda^{-1}\widetilde{\mathrm{curl}}\phi_u, \tag{3.1}$$

with

$$\widehat{\mathcal{G}}(t,\xi) = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} \\ \mathcal{G}^{21} & \mathcal{G}^{22} \end{pmatrix} = \begin{pmatrix} \widetilde{K}(t,\xi) & -i\bar{\rho}L(t,\xi)\xi^\top \\ \frac{\tilde{P}}{\bar{\rho}}L(t,\xi)|\xi| & iK(t,\xi)\frac{\xi^\top}{|\xi|} \end{pmatrix}.$$

Here $K(t,\eta) = K_1(t,\eta) + K_2(t,\eta)$ and $L(t,\xi) = L_1(t,\xi) + L_2(t,\xi)$ with

$$K_1(t,\eta) = \frac{\lambda_+(\eta)e^{\lambda_+(\eta)t}}{\lambda_+(\eta)-\lambda_-(\eta)}, \qquad K_2(t,\eta) = -\frac{\lambda_-(\eta)e^{\lambda_-(\eta)t}}{\lambda_+(\eta)-\lambda_-(\eta)},$$
$$L_1(t,\xi) = -\frac{e^{\lambda_+(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}, \qquad L_2(t,\xi) = \frac{e^{\lambda_-(\xi)t}}{\lambda_+(\xi)-\lambda_-(\xi)}.$$

The following facts can be easily verified: for $|\xi| \gg 1$,

($\mathbf{b}_1$) $\big\{\lambda_-(\xi),\ \lambda_+(\xi),\ \lambda_+(\xi)-\lambda_-(\xi)\big\}$ behaves like $\big\{-\nu|\xi|^2,\ -\tilde{P}\nu^{-1},\ \nu|\xi|^2\big\}$,

($\mathbf{b}_2$) $\big\{K_1(t,\xi),\ K_2(t,\xi)\big\}$ behaves like $\big\{-\tilde{P}(\nu|\xi|)^{-2}e^{-\tilde{P}\nu^{-1}t},\ e^{-\nu|\xi|^2t}\big\}$,

($\mathbf{b}_3$) $\big\{L_1(t,\xi),\ L_2(t,\xi)\big\}$ behaves like $\big\{-(\nu|\xi|^2)^{-1}e^{-\tilde{P}\nu^{-1}t},\ (\nu|\xi|^2)^{-1}e^{-\nu|\xi|^2t}\big\}$.

The second derivative (ρ'', u'') of (ρ, u) with respect to δ at $\delta=0$ satisfies

$$\begin{cases} \partial_t\rho'' + 2\rho'\mathrm{div}u' + \bar{\rho}\mathrm{div}u'' + 2\nabla\rho' u' = 0, \\ \partial_t u'' - \bar{\mu}\Delta u'' - (\bar{\lambda}+\bar{\mu})\nabla\mathrm{div}u'' + \displaystyle\sum_{J=1}^{4} H_J = 0, \\ (\rho_0'', u_0'') = (0,0), \end{cases}$$

where

$$\sum_{J=1}^{4} H_J \triangleq 2\bar{\rho}^{-1}\rho'\partial_t u' + 2u'\cdot\nabla u' + 2\bar{\rho}^{-1}P''(\bar{\rho})\rho'\nabla\rho' + \bar{\rho}^{-1}P'(\bar{\rho})\nabla\rho''.$$

Set $\Omega'' = \Lambda^{-1}\widetilde{\mathrm{curl}}u''$, and it satisfies

$$\partial_t\Omega'' - \bar{\mu}\Delta\Omega'' = -\Lambda^{-1}\widetilde{\mathrm{curl}}\sum_{J=1}^{4} H_J.$$

To prove Theorem 1.2, it suffices to prove the unboundedness in $\mathcal{S}'(\mathbb{R}^3)$ of the incompressible part $-\Lambda^{-1}\mathrm{div}\Omega''$ of u'', which satisfies

$$-\Lambda^{-1}\mathrm{div}\Omega''(x,t) = \sum_{J=1}^{4}\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}\int_0^t e^{\bar{\mu}\Delta(t-\tau)}H_J d\tau \triangleq \sum_{J=1}^{4}\mathfrak{H}_J. \tag{3.2}$$

The initial data ϕ_u will be chosen differently with (2.6) and is given by

$$\widehat{\phi_u}(\xi) = \sum_{k=N_0}^{N}\tilde{\alpha}_k 2^{(1-\frac{3}{p})k}\big(\phi(\xi-2^k\tilde{e}) + \phi(\xi+2^k\tilde{e}),\ i\phi(\xi-2^k\tilde{e}) - i\phi(\xi+2^k\tilde{e}),\ 0\big),$$

here N_0 is an integer satisfying

$$N_0 = \max\,(20,\ 4\nu^{-1}\tilde{P}^{\frac{1}{2}}),$$

and N is a large enough integer, $\tilde{e} = (1,1,0)$, Set $\{\tilde{\alpha}_k\}$ is a series belonging to ℓ^1, and ϕ defined as in (2.6). Take $\phi_\rho \in \mathcal{S}(\mathbb{R}^3)$ such that $\widehat{\phi}_\rho(\xi) = \phi(\xi)$. It is easy to check that ϕ_u is a real valued function and uniformly bounded in $\dot{B}^{\frac{3}{p}-1}_{p,1}$.

In what follows, let us fix $\xi_0 = (0, 1/10, 1/10)$ and always assume $\xi \in B(\xi_0, \epsilon)$ and $t = \frac{1}{2^{20}}$.

- Due to $\widetilde{\mathrm{curl}}\,H_3 = \widetilde{\mathrm{curl}}\,H_4 = 0$, we get

$$\mathfrak{H}_3 = \mathfrak{H}_4 = 0. \tag{3.3}$$

- The estimate of $\mathfrak{H}_2$

By (3.1), we have

$$\begin{aligned} u' &= -\Lambda^{-1}\nabla h' - \Lambda^{-1}\mathrm{div}\Omega' \\ &= -\Lambda^{-1}\nabla(\mathcal{G}^{21}\phi_\rho + \mathcal{G}^{22}\phi_u) - \Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar{\mu}\Delta t}\phi_u. \end{aligned}$$

Hence,

$$u' \cdot \nabla u' = H_{21} + \cdots + H_{26},$$

where

$$\begin{aligned}
H_{21} =&\Lambda^{-1}\nabla\mathcal{G}^{21}\phi_\rho \cdot \nabla\Lambda^{-1}\nabla\mathcal{G}^{21}\phi_\rho,\\
H_{22} =&\Lambda^{-1}\nabla\mathcal{G}^{21}\phi_\rho \cdot \nabla\Lambda^{-1}(\nabla\mathcal{G}^{22}\phi_u + \Lambda^{-1}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u)\\
&+ \Lambda^{-1}(\nabla\mathcal{G}^{22}\phi_u + \Lambda^{-1}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u) \cdot \nabla\Lambda^{-1}\nabla\mathcal{G}^{21}\phi_\rho,\\
H_{23} =&\Lambda^{-1}\nabla\mathcal{G}^{22}\phi_u \cdot \nabla\Lambda^{-1}\nabla\mathcal{G}^{22}\phi_u,\\
H_{24} =&\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u \cdot \nabla\Lambda^{-1}\nabla\mathcal{G}^{22}\phi_u,\\
H_{25} =&\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u \cdot \nabla\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u,\\
H_{26} =&\Lambda^{-1}\nabla\mathcal{G}^{22}\phi_u \cdot \nabla\Lambda^{-2}\mathrm{div}\mathrm{curl}e^{\bar\mu\Delta t}\phi_u.
\end{aligned}$$

Due to the choice of ϕ_ρ and ϕ_u, we find that

$$\widehat{H_{22}}(t,\xi) = 0 \quad \text{for } \xi \in B(\xi_0, \epsilon), \quad H_{21} \in L^2(\mathbb{R}^3). \tag{3.4}$$

Noticing that

$$H_{23} = \partial_\ell(\Lambda^{-1}\mathcal{G}^{22}\phi_u)\partial_\ell\partial_j(\Lambda^{-1}\mathcal{G}^{22}\phi_u) = \frac{1}{2}\partial_j(\partial_\ell\Lambda^{-1}\mathcal{G}^{22}\phi_u)^2,$$

it follows that

$$\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}\int_0^t e^{\bar\mu\Delta(t-\tau)}H_{23}(\tau)d\tau = 0. \tag{3.5}$$

Thanks to $\mathrm{div}(\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u) = 0$, we get

$$\begin{aligned}
H_{24} =&\mathrm{div}(\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u \otimes \Lambda^{-1}\nabla\mathcal{G}^{22}\phi_u),\\
H_{25} =&\mathrm{div}(\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u \otimes \Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar\mu\Delta t}\phi_u).
\end{aligned}$$

Let $\mathfrak{H}_{24}$ and $\mathfrak{H}_{25}$ be the term corresponding to H_{24} and H_{25} in $\mathfrak{H}_2$ respectively, whose Fourier transform is given by

$$\begin{aligned}
\big(\widehat{\mathfrak{H}}_{24}\big)_j(t,\xi) = &-\frac{e^{-\bar\mu|\xi|^2 t}}{|\xi|^2}\int_0^t\!\!\int_{\mathbb{R}^3}\frac{e^{(\bar\mu|\xi|^2-\bar\mu|\xi-\eta|^2)\tau}}{|\xi-\eta|^2|\eta|}(\xi_{j'}\xi_j\eta_{j'} - \xi_{j'}\xi_{j'}\eta_j)\widehat{\mathcal{G}^{22}\phi_u}(\eta)\\
&\times \xi_{\ell'}(\xi-\eta)_\ell\big((\xi-\eta)_{\ell'}\widehat{\phi}_u^\ell(\xi-\eta) - (\xi-\eta)_\ell\widehat{\phi}_u^{\ell'}(\xi-\eta)\big)d\eta d\tau
\end{aligned}$$

for $j = 1, 2, 3$. Recall that

$$\widehat{\mathcal{G}^{22}\phi_u}(\eta) = i\frac{K(\tau,\eta)}{|\eta|}\big(\eta_1\widehat{\phi}_u^1(\eta) + \eta_2\widehat{\phi}_u^2(\eta)\big),$$

hence for $j = 1$,

$$\begin{aligned}
\big(\widehat{\mathfrak{H}}_{24}\big)_1(t,\xi) = &-i\frac{e^{-\bar\mu|\xi|^2 t}}{|\xi|^2}\int_0^t\!\!\int_{\mathbb{R}^3}\frac{e^{(\bar\mu|\xi|^2-\bar\mu|\xi-\eta|^2)\tau}}{|\xi-\eta|^2|\eta|^2}\big(\xi_1(\xi_2\eta_2 + \xi_3\eta_3) - (\xi_2^2 + \xi_3^2)\eta_1\big)\\
&\times (K_1 + K_2)(\tau,\eta)\big(\eta_1\widehat{\phi}_u^1(\eta) + \eta_2\widehat{\phi}_u^2(\eta)\big)\big(\widehat{\phi}_u^1(\xi-\eta)\Box_1 + \widehat{\phi}_u^2(\xi-\eta)\Box_2\big)d\eta d\tau,
\end{aligned}$$

where

$$\begin{aligned}
\Box_1 &\triangleq (\xi-\eta)_2(\xi_1\eta_2 - \eta_1\xi_2) + (\xi-\eta)_3(\xi_1\eta_3 - \xi_3\eta_1),\\
\Box_2 &\triangleq (\xi-\eta)_1(\eta_1\xi_2 - \xi_1\eta_2) + (\xi-\eta)_3(\xi_2\eta_3 - \xi_3\eta_2).
\end{aligned}$$

Then we obtain

$$\begin{aligned}\big(\widehat{\mathfrak{H}}_{24}\big)_1 = -i\frac{e^{-\bar\mu|\xi|^2 t}}{|\xi|^2}\sum_{k=N_0}^{N}\int_{\mathbb{R}^3}\frac{\tilde\alpha_k^2 2^{2k(1-\frac{3}{p})}}{\lambda_+(\eta)-\lambda_-(\eta)}\bigg(\frac{(e^{(\bar\mu|\xi|^2-\bar\mu|\xi-\eta|^2+\lambda_+(\eta))t}-1)\lambda_+(\eta)}{\bar\mu|\xi|^2-\bar\mu|\xi-\eta|^2+\lambda_+(\eta)}\\ -\frac{(e^{(\bar\mu|\xi|^2-\bar\mu|\xi-\eta|^2+\lambda_-(\eta))t}-1)\lambda_-(\eta)}{\bar\mu|\xi|^2-\bar\mu|\xi-\eta|^2+\lambda_-(\eta)}\bigg)\frac{\mathcal{O}(\phi,\eta,\xi,k)}{|\xi-\eta|^2|\eta|^2}d\eta,\end{aligned}$$

where

$$\begin{aligned}\mathcal{O}(\phi,\eta,\xi,k) =&(\eta_1\Box_1+\eta_2\Box_2-\eta_1\Box_2 i+\eta_2\Box_1 i)\phi(\eta-2^k\tilde e)\phi(\xi-\eta+2^k\tilde e)\\ &+(\eta_1\Box_1+\eta_2\Box_2+\eta_1\Box_2 i-\eta_2\Box_1 i)\phi(\eta+2^k\tilde e)\phi(\xi-\eta-2^k\tilde e).\end{aligned}$$

Using $(\mathbf{b}_1)$-$(\mathbf{b}_2)$ and the choice of ϕ_u,ξ_0, we have

$$\big|\big(\mathrm{Re}(\widehat{\mathfrak{H}}_{24})\big)_1(t,\xi)\big| \le C\sum_{k=N_0}^{N}\tilde\alpha_k^2 2^{-\frac{6}{p}k}. \tag{3.6}$$

Similarly, we can obtain

$$\big|\big(\mathrm{Re}(\widehat{\mathfrak{H}}_{25})\big)_1(t,\xi)\big| \le C\sum_{k=N_0}^{N}\tilde\alpha_k^2 2^{-\frac{6}{p}k}. \tag{3.7}$$

Let $\mathfrak{H}_{26}$ be the term corresponding to H_{26} in $\mathfrak{H}_2$, whose Fourier transform is given by

$$\begin{aligned}\big(\widehat{\mathfrak{H}}_{26}\big)_j(t,\xi) = -\frac{e^{-\bar\mu|\xi|^2 t}}{|\xi|^2}\int_0^t\!\!\int_{\mathbb{R}^3}&\frac{e^{(\bar\mu|\xi|^2-\bar\mu|\eta|^2)\tau}}{|\xi-\eta||\eta|^2}(\xi-\eta)_{\ell'}\eta_{\ell'}\widehat{\mathcal{G}^{22}\phi_u}(\xi-\eta)\\ &\times\Big[\xi_{j'}\xi_j(\eta_\ell\eta_{j'}\widehat{\phi}_u^\ell(\eta)-\eta_\ell\eta_\ell\widehat{\phi}_u^{j'}(\eta))\\ &-\xi_{j'}\xi_{j'}(\eta_\ell\eta_j\widehat{\phi}_u^\ell(\eta)-\eta_\ell\eta_\ell\widehat{\phi}_u^{j}(\eta))\Big]d\eta d\tau,\qquad j=1,2,3.\end{aligned}$$

For $j=1$, the term in the square brackets of the above integral equals to

$$\begin{aligned}&\widehat{\phi}_u^1(\eta)\big(\eta_1\eta_2\xi_1\xi_2+\eta_1\eta_3\xi_1\xi_3+\eta_2^2(\xi_2^2+\xi_3^2)+\eta_3^2(\xi_2^2+\xi_3^2)\big)\\ &\quad+\widehat{\phi}_u^2(\eta)\big(\eta_2\eta_3\xi_1\xi_3-\eta_3^2\xi_1\xi_2-\eta_1^2\xi_1\xi_2-\eta_1\eta_2(\xi_2^2+\xi_3^2)\big)\\ &\triangleq\widehat{\phi}_u^1(\eta)(O_1(\eta,\xi)+o_1(\eta,\xi))+\widehat{\phi}_u^2(\eta)(O_2(\eta,\xi)+o_2(\eta,\xi))\end{aligned}$$

with

$$\begin{aligned}O_1(\eta,\xi)&=\eta_1\eta_2\xi_1\xi_2+\eta_2^2(\xi_2^2+\xi_3^2),\quad o_1(\eta,\xi)=\eta_1\eta_3\xi_1\xi_3+\eta_3^2(\xi_2^2+\xi_3^2),\\ O_2(\eta,\xi)&=-\eta_1^2\xi_1\xi_2-\eta_1\eta_2(\xi_2^2+\xi_3^2),\quad o_1(\eta,\xi)=\eta_2\eta_3\xi_1\xi_3-\eta_3^2\xi_1\xi_2.\end{aligned}$$

On the other hand, we have

$$\widehat{\mathcal{G}^{22}\phi_u}(\xi-\eta)=i\frac{K(\tau,\xi-\eta)}{|\xi-\eta|}\big((\xi-\eta)_1\widehat{\phi}_u^1(\xi-\eta)+(\xi-\eta)_2\widehat{\phi}_u^2(\xi-\eta)\big).$$

Then we find that

$$\begin{aligned}\big(\widehat{\mathfrak{H}}_{26}\big)_1 = -i\frac{e^{-\bar\mu|\xi|^2 t}}{|\xi|^2}\int_0^t\!\!\int_{\mathbb{R}^3}&\frac{e^{\bar\mu(|\xi|^2-|\eta|^2)\tau}}{|\xi-\eta|^2|\eta|^2}(K_1+K_2)(\tau,\xi-\eta)(\xi-\eta)_{\ell'}\eta_{\ell'}\\ &\times\sum_{k=N_0}^{N}\tilde\alpha_k^2 2^{2k(1-\frac{3}{p})}\widetilde{\mathcal{O}}(\phi,\eta,\xi,k)d\eta d\tau,\end{aligned}$$

where

$$\begin{aligned}\widetilde{\mathcal{O}}(\phi,\eta,\xi,k) =&\big\{(\xi-\eta)_1(O_1(\eta,\xi)+o_1(\eta,\xi))+(\xi-\eta)_2(O_2(\eta,\xi)+o_2(\eta,\xi))\\&-i(\xi-\eta)_1(O_2(\eta,\xi)+o_2(\eta,\xi))+i(\xi-\eta)_2(O_1(\eta,\xi)+o_1(\eta,\xi))\big\}\\&\times\phi(\eta-2^k\tilde{e})\phi(\xi-\eta+2^k\tilde{e})+\big\{(\xi-\eta)_1(O_1(\eta,\xi)+o_1(\eta,\xi))\\&+(\xi-\eta)_2(O_2(\eta,\xi)+o_2(\eta,\xi))+i(\xi-\eta)_1(O_2(\eta,\xi)+o_2(\eta,\xi))\\&-i(\xi-\eta)_2(O_1(\eta,\xi)+o_1(\eta,\xi))\big\}\phi(\eta+2^k\tilde{e})\phi(\xi-\eta-2^k\tilde{e}).\end{aligned}$$

Then we have

$$\begin{aligned}\big(\widehat{\mathfrak{H}}_{26}\big)_1 =&-i\frac{e^{-\bar{\mu}|\xi|^2t}}{|\xi|^2}\int_{\mathbb{R}^3}\sum_{k=N_0}^{N}\frac{\tilde{\alpha}_k^2 2^{2k(1-\frac{3}{p})}}{\lambda_+(\xi-\eta)-\lambda_-(\xi-\eta)}\bigg(\frac{(\exp_+(t,\xi,\eta)-1)\lambda_+(\xi-\eta)}{\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_+(\xi-\eta)}\\&-\frac{(\exp_-(t,\xi,\eta)-1)\lambda_-(\xi-\eta)}{\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_-(\xi-\eta)}\bigg)\frac{(\xi-\eta)_\ell\eta_\ell}{|\xi-\eta|^2|\eta|^2}\widetilde{\mathcal{O}}(\phi,\eta,\xi,k)d\eta,\end{aligned}$$

where

$$\exp_+(t,\xi,\eta)=e^{(\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_+(\xi-\eta))t},\ \ \exp_-(t,\xi,\eta)=e^{(\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_-(\xi-\eta))t}.$$

Due to the choice of ξ_0 and ϕ_u and using ($\mathbf{b}_1$)-($\mathbf{b}_2$), we can get by some tedious computations that

$$\big(\mathrm{Re}(\widehat{\mathfrak{H}}_{26})\big)_1(t,\xi)\ge c_0\sum_{k=N_0}^{N}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}. \tag{3.8}$$

Summing up (3.4)-(3.8), we conclude that

$$\big(\mathrm{Re}(\widehat{\mathfrak{H}}_{2})\big)_1(t,\xi)\ge \frac{3}{4}c_0\sum_{k=N_0}^{N}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}. \tag{3.9}$$

- The estimate of $\mathfrak{H}_1$

Recall that

$$\rho'\partial_t u'=\rho'\big(\bar{\mu}\Delta u'+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u'\big)-\frac{P'(\bar{\rho})}{\bar{\rho}}\rho'\nabla\rho'\triangleq H_{11}+H_{12}.$$

In light of $\widetilde{\mathrm{curl}}H_{12}=0$, we get

$$\mathfrak{H}_{12}=0. \tag{3.10}$$

By (3.1),

$$\begin{aligned}H_{11}=&\bar{\nu}(\mathcal{G}^{11}\phi_\rho+\mathcal{G}^{12}\phi_u)\Lambda\nabla(\mathcal{G}^{21}\phi_\rho+\mathcal{G}^{22}\phi_u)\\&+\bar{\mu}(\mathcal{G}^{11}\phi_\rho+\mathcal{G}^{12}\phi_u)\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar{\mu}\Delta t}\phi_u.\end{aligned}$$

Obviously, the Fourier transform of the following three functions

$$\mathcal{G}^{11}\phi_\rho\nabla\Lambda\mathcal{G}^{12}\phi_u,\ \ \mathcal{G}^{12}\phi_u\Lambda\nabla\mathcal{G}^{21}\phi_\rho,\ \ \mathcal{G}^{11}\phi_\rho\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar{\mu}\Delta t}\phi_u.$$

vanishes on $B(\xi_0,\epsilon)$. Due to the choice of ϕ_ρ, we infer that

$$\Lambda^{-2}\mathrm{div}\widetilde{\mathrm{curl}}\int_0^t e^{\bar{\mu}\Delta(t-\tau)}\mathcal{G}^{11}\phi_\rho\nabla\Lambda\mathcal{G}^{21}\phi_\rho(\tau,x)d\tau\in L^2(\mathbb{R}^3). \tag{3.11}$$

Let $\mathfrak{H}_{111}$ and $\mathfrak{H}_{112}$ be the terms in $\mathfrak{H}_1$ corresponding to

$$\mathcal{G}^{12}\phi_u\Lambda\nabla\mathcal{G}^{22}\phi_u\quad\text{and}\quad\mathcal{G}^{12}\phi_u\mathrm{div}\widetilde{\mathrm{curl}}e^{\bar{\mu}\Delta t}\phi_u.$$

We have

$$(\widehat{\mathfrak{H}}_{112})_j = \frac{e^{-\bar{\mu}|\xi|^2 t}}{|\xi|^2}\int_0^t\!\!\int_{\mathbb{R}^3} e^{\bar{\mu}(|\xi|^2-|\eta|^2)\tau}\widehat{\mathcal{G}^{12}\phi_u}(\xi-\eta)\big[\xi_{j'}\xi_j(\eta_\ell\eta_{j'}\widehat{\phi}_u^\ell(\eta)-\eta_\ell\eta_\ell\widehat{\phi}_u^{j'}(\eta)) - \xi_{j'}\xi_{j'}(\eta_\ell\eta_j\widehat{\phi}_u^\ell(\eta)-\eta_\ell\eta_\ell\widehat{\phi}_u^j(\eta))\big]d\eta d\tau.$$

Especially for $j=1$, we find

$$\big(\widehat{\mathfrak{H}}_{112}\big)_1 = -i\bar{\rho}\frac{e^{-\bar{\mu}|\xi|^2 t}}{|\xi|^2}\sum_{k=N_0}^{N}\tilde{\alpha}_k^2\int_0^t\!\!\int_{\mathbb{R}^3} 2^{2k(1-\frac{3}{p})}e^{\bar{\mu}(|\xi|^2-|\eta|^2)\tau} \times (L_1+L_2)(\tau,\xi-\eta)\widetilde{\mathcal{O}}(\phi,\eta,\xi,k)d\eta d\tau.$$

The integral part of the right hand side equals to

$$\int_{\mathbb{R}^3} 2^{2k(1-\frac{3}{p})}\Big(\frac{\exp_-(t,\xi,\eta)-1}{\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_-(\xi-\eta)} - \frac{\exp_+(t,\xi,\eta)-1}{\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_+(\xi-\eta)}\Big)\widetilde{\mathcal{O}}(\phi,\eta,\xi,k)d\eta.$$

Using $(\mathbf{b}_1),(\mathbf{b}_3)$ and the choice of ϕ_u and ξ, one can verify that

$$2^k\int_{\mathbb{R}^3}\Big(\frac{\exp_-(t,\xi,\eta)-1}{\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_-(\xi-\eta)} - \frac{\exp_+(t,\xi,\eta)-1}{\bar{\mu}|\xi|^2-\bar{\mu}|\eta|^2+\lambda_+(\xi-\eta)}\Big)\widetilde{\mathcal{O}}(\phi,\eta,\xi,k)d\eta \to 0$$

if $k\to+\infty$, which means that for any $\delta>0$, there exists a large integer N_1 such that for $k\geq N_1$,

$$\Big|2^k\int_0^t\!\!\int_{\mathbb{R}^3} e^{\bar{\mu}(|\xi|^2-|\eta|^2)\tau}(L_1+L_2)(\tau,\xi-\eta)\widetilde{\mathcal{O}}(\phi,\eta,\xi,k)d\eta d\tau\Big| \leq \delta.$$

This in turn implies that

$$|\big(\mathrm{Re}(\widehat{\mathfrak{H}}_{112})\big)_1(t,\xi)| \leq \delta\sum_{k\geq N_1}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k} + C\sum_{k=N_0}^{N_1}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}(\widehat{\mathfrak{H}}_{111})_1 &= -i\frac{e^{-\bar{\mu}|\xi|^2 t}}{|\xi|^2}\int_0^t\!\!\int_{\mathbb{R}^3} e^{\bar{\mu}|\xi|^2\tau}\widehat{\mathcal{G}^{12}\phi_u}(\xi-\eta)\widehat{\mathcal{G}^{22}\phi_u}(\eta)|\eta|\big(\xi_{j'}\xi_1\eta_{j'}-\xi_{j'}\xi_{j'}\eta_1\big)d\eta d\tau\\ &= -i\bar{\rho}\frac{e^{-\bar{\mu}|\xi|^2 t}}{|\xi|^2}\sum_{k=10}^{N}\tilde{\alpha}_k^2\int_0^t\!\!\int_{\mathbb{R}^3} 2^{2k(1-\frac{3}{p})}e^{\bar{\mu}|\xi|^2\tau}L(\tau,\xi-\eta)K(\tau,\eta)\\ &\qquad\times\big(\xi_1(\xi_2\eta_2+\xi_3\eta_3)-(\xi_2^2+\xi_3^2)\eta_1\big)\widetilde{\widetilde{\mathcal{O}}}(\phi,\eta,\xi,k)d\eta d\tau,\end{aligned}$$

where

$$\begin{aligned}\widetilde{\widetilde{\mathcal{O}}}(\phi,\eta,\xi,k) =&[(\xi-\eta)_1\eta_1+(\xi-\eta)_2\eta_2-i(\xi-\eta)_1\eta_2+i(\xi-\eta)_2\eta_1]\\ &\times\phi(\xi-\eta-2^k\tilde{e})\phi(\eta+2^k\tilde{e})+[(\xi-\eta)_1\eta_1+(\xi-\eta)_2\eta_2\\ &+i(\xi-\eta)_1\eta_2-i(\xi-\eta)_2\eta_1]\phi(\xi-\eta+2^k\tilde{e})\phi(\eta-2^k\tilde{e}).\end{aligned}$$

Recalling the definition of $K(\tau,\eta)$ and $L(\tau,\xi-\eta)$, we get

$$\int_0^t e^{\bar{\mu}|\xi|^2\tau}K(\tau,\eta)L(\tau,\xi-\eta)d\tau=(\lambda_+(\xi-\eta)-\lambda_-(\xi-\eta))^{-1}(\lambda_+(\eta)-\lambda_-(\eta))^{-1}$$
$$\Big(\frac{-\lambda_+(\eta)(e^{(\bar{\mu}|\xi|^2+\lambda_+(\xi-\eta)+\lambda_+(\eta))t}-1)}{\bar{\mu}|\xi|^2+\lambda_+(\xi-\eta)+\lambda_+(\eta)}+\frac{\lambda_-(\eta)(e^{(\bar{\mu}|\xi|^2+\lambda_+(\xi-\eta)+\lambda_-(\eta))t}-1)}{\bar{\mu}|\xi|^2+\lambda_+(\xi-\eta)+\lambda_-(\eta)}$$
$$\frac{\lambda_+(\eta)(e^{(\bar{\mu}|\xi|^2+\lambda_-(\xi-\eta)+\lambda_+(\eta))t}-1)}{\bar{\mu}|\xi|^2+\lambda_-(\xi-\eta)+\lambda_+(\eta)}-\frac{\lambda_-(\eta)(e^{(\bar{\mu}|\xi|^2+\lambda_-(\xi-\eta)+\lambda_-(\eta))t}-1)}{\bar{\mu}|\xi|^2+\lambda_-(\xi-\eta)+\lambda_-(\eta)}\Big).$$

By the same argument as leading to the (3.12), we find that for $\delta>0$, there exists a large integer N_2 such that for $k\ge N_2$,

$$|\big(\mathrm{Re}(\widehat{\mathfrak{H}}_{111})\big)_1(t,\xi)|\le\delta\sum_{k\ge N_2}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}+C\sum_{k=N_0}^{N_2}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}. \tag{3.13}$$

Combining (3.10)-(3.13), and choosing $\delta=\frac{c_0}{8}$, we infer that

$$|\big(\mathrm{Re}(\widehat{\mathfrak{H}}_1)\big)_1(t,\xi)|\le\frac{c_0}{4}\sum_{k\ge N_2}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}+2^{(1-\frac{6}{p})\max(N_1,N_2)}. \tag{3.14}$$

Collecting (3.2), (3.3), (3.9) and (3.14), we conclude that

$$\big(\mathrm{Re}(\widehat{\Lambda^{-1}\mathrm{div}\Omega''})\big)_1(t,\xi)\ge\frac{c_0}{4}\sum_{k\ge N_0}\tilde{\alpha}_k^2 2^{(1-\frac{6}{p})k}.$$

This is enough to conclude that $\Lambda^{-1}\mathrm{div}\Omega''$ is not bounded in $\mathcal{S}'(\mathbb{R}^3)$ if $p>6$. This in turn implies that u'' is not bounded in $\mathcal{S}'(\mathbb{R}^3)$.

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